Pattern Recognition

Introduction to Probability

Bayes Decision Theory

Review

- Features: These are measurable quantities obtained from the patterns, and the classification task is based on their respective values.

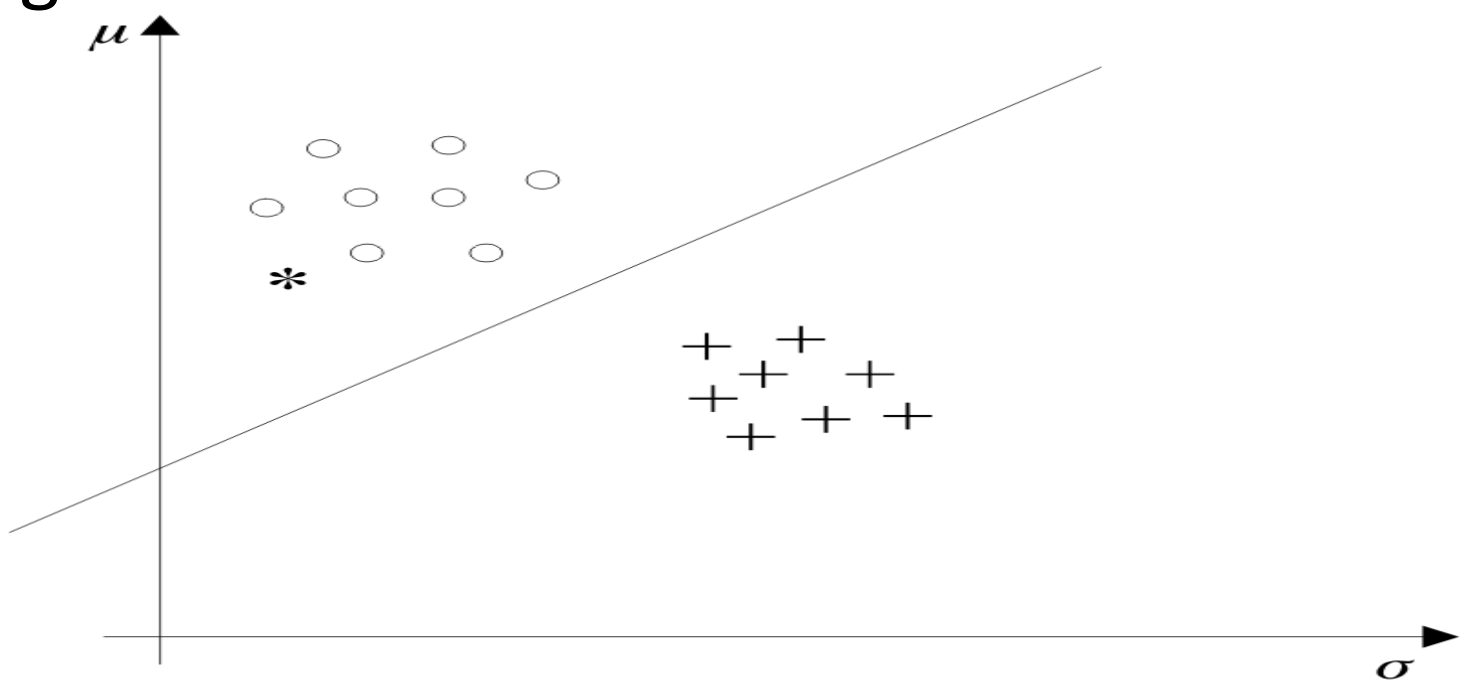
- Feature vectors: A number of feature values $<x_1,x_2,..,x_n>$ constitute the feature vector

- Feature vectors are treated as random vectors.

- A random variable is a variable whose value is subject to variations due to chance and it can take on a set of possible different values, each with an associated probability.

Review

- The classifier consists of a set of functions, whose values, computed at $\underline{X}'$ determine the class to which the corresponding pattern belongs



Classification Based on Bayes Decision Theory

- Statistical nature of feature vectors

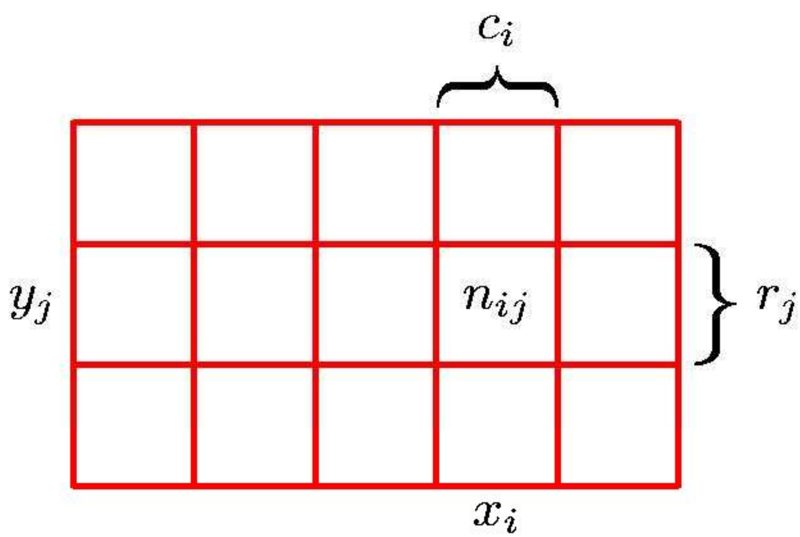
$$\underline{x} = \left[x_1, x_2, \ldots, x_l\right]^T$$

- Assign the pattern represented by feature vector $\underline{x}$ to the most probable of the available classes

$$\omega_1, \omega_2, \ldots, \omega_M$$

That is $\underline{x} \rightarrow \omega_i : \underbrace{P(\omega_i | \underline{x})}_{\text{maximum}}$

Probability Theory



Marginal Probability

$$p(X = x_i) = \frac{c_i}{N}.$$

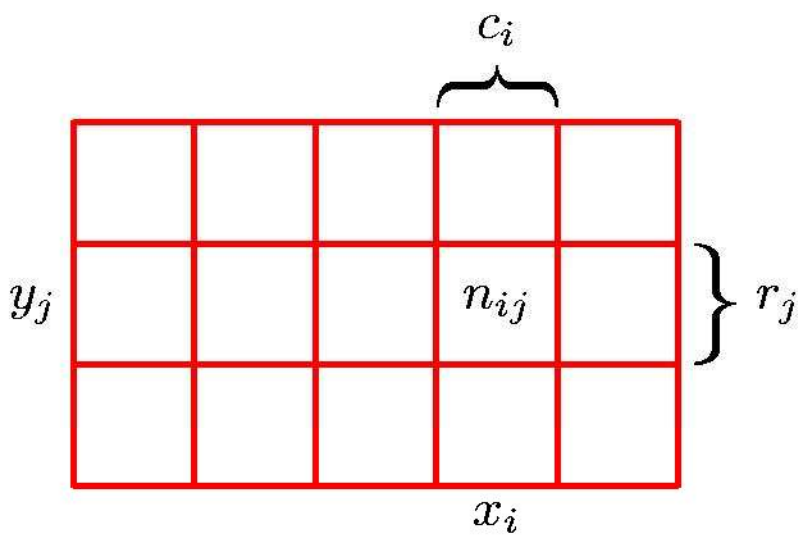
Joint Probability

$$p(X = x_i, Y = y_j) = \frac{n_{ij}}{N}$$

Conditional Probability

$$p(Y = y_j | X = x_i) = \frac{n_{ij}}{c_i}$$

Probability Theory



Sum Rule

$$p(X = x_i) = \frac{c_i}{N} = \frac{1}{N}\sum_{j=1}^{L} n_{ij}$$

$$= \sum_{j=1}^{L} p(X = x_i, Y = y_j)$$

Product Rule

$$\begin{aligned} p(X = x_i, Y = y_j) &= \frac{n_{ij}}{N} = \frac{n_{ij}}{c_i} \cdot \frac{c_i}{N} \\ &= p(Y = y_j | X = x_i) p(X = x_i) \end{aligned}$$

The Rules of Probability

Sum Rule $$p(X) = \sum_Y p(X, Y)$$

Product Rule $$p(X, Y) = p(Y|X)p(X)$$

Example

- A jar contains 10 black and 10 white marbles. Two marbles are chosen without replacement. What is the probability of drawing a white marble and then a black one?

  $P(x=white\ ,\ y=black) = P(y|x).P(x) = (10/19)(1/2)$

Example

- A jar contains black and white marbles. Two marbles are chosen without replacement. The probability of selecting a black marble and then a white marble is 0.34, and the probability of selecting a black marble on the first draw is 0.47. What is the probability of selecting a white marble on the second draw, given that the first marble drawn was black?
- $P(x=\text{black}, y=\text{white})=0.34=p(y|x).p(x)$
- $P(y|x)=0.72$

Independent probability

- If two events, A and B are independent then the joint probability is

$$P(A \text{ and } B) = P(A) \,.\, P(B)$$

a-posteriori probabilities

- The a-posteriori probability is the probability of the parameter T given the evidence p(x|T)
- Example: Suppose there is a box of marbles with 60% of them being square and 40% being round. The round marbles are either black or white in equal numbers; the square marbles are all black. An observer sees a (random) marble from a distance; all the observer can see is that this marble is black. What is the probability this marble is round?

a-posteriori probabilities

- a-priori probabilities $P(\omega_1), P(\omega_2)..., P(\omega_M)$

  Then

$$p(\underline{x}|\omega_i), i = 1,2...M$$

- This is also known as the likelihood $\underline{x}$ with respect to ω_i .

Bayes' Theorem

$$p(Y|X) = \frac{p(X|Y)p(Y)}{p(X)}$$

$$p(X) = \sum_{Y} p(X|Y)p(Y)$$

a-posterior $\propto$ likelihood × prior

Example

- Suppose there is a box of marbles with 60% of them being square and 40% being round. The round marbles are either black or white in equal numbers; the square marbles are all black. An observer sees a (random) marble from a distance; all the observer can see is that this marble is black. What is the probability this marble is round?
- $P(\text{round})=0.4$
- $P(\text{black}|\text{round})=0.5$
- $P(\text{black})=p(\text{black}|\text{round}).p(\text{round}) + p(\text{black}|\text{square}).p(\text{square}) = 0.5*0.4+1*0.6 = 0.8$
- $P(\text{round}|\text{black})=(p(\text{black}|\text{round}).p(\text{round}))/P(\text{black})$
- $0.5 \text{x} 0.4/0.8=0.25$

The Bayes classification rule (for two classes M=2)

- Given $\underline{x}$ classify it according to the rule

$$\boxed{\begin{aligned} &If\ P(\omega_1|\underline{x}) > P(\omega_2|\underline{x})\ \ \underline{x} \rightarrow \omega_1 \\ &If\ P(\omega_2|\underline{x}) > P(\omega_1|\underline{x})\ \ \underline{x} \rightarrow \omega_2 \end{aligned}}$$

- Equivalently: classify $\underline{x}$ according to the rule

$$\boxed{p(\underline{x}|\omega_1)P(\omega_1)(><)p(\underline{x}|\omega_2)P(\omega_2)}$$

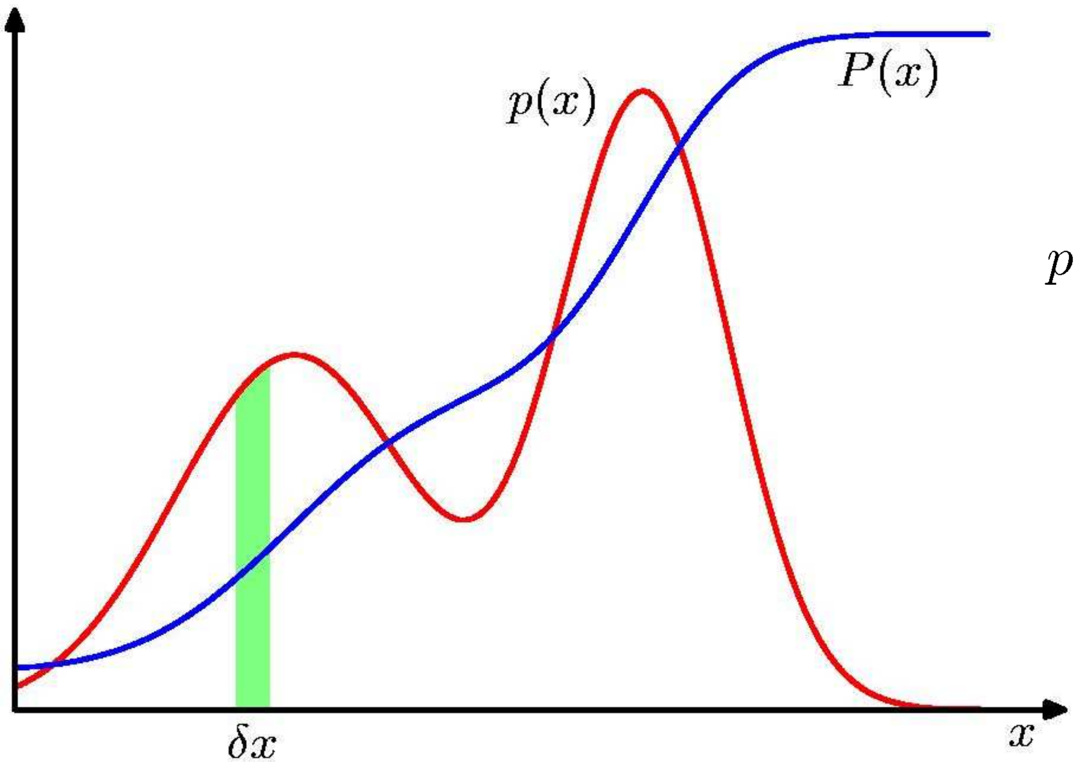
- For equiprobable classes the test becomes

$$\boxed{p(\underline{x}|\omega_1)(><)P(\underline{x}|\omega_2)}$$

Probability Density Function

- A **probability density function** (**pdf**), or **density** of a continuous random variable, is a function that describes the relative likelihood for this random variable to take on a given value.

Probability Densities



$$p(x \in (a, b)) = \int_a^b p(x)\,\mathrm{d}x$$

$$P(z) = \int_{-\infty}^{z} p(x)\,\mathrm{d}x$$

$$p(x) \geqslant 0 \qquad \int_{-\infty}^{\infty} p(x)\,\mathrm{d}x = 1$$

Cumulative Distribution Function

- **Cumulative distribution function** (**CDF**), or just **distribution function**, describes the probability that a real-valued random variable *X* with a given probability distribution will be found at a value less than or equal to *x*.

Expectations

$$\mathbb{E}[f] = \sum_x p(x) f(x)$$

$$\mathbb{E}[f] = \int p(x) f(x) \, \mathrm{d}x$$

$$\mathbb{E}_x[f|y] = \sum_x p(x|y) f(x)$$

Conditional Expectation
(discrete)

$$\mathbb{E}[f] \simeq \frac{1}{N} \sum_{n=1}^{N} f(x_n)$$

Approximate Expectation
(discrete and continuous)